Green functions for killed random walks in the Weyl chamber of Sp(4)

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Abstract

We consider a family of random walks killed at the boundary of the Weyl chamber of the dual of Sp(4), which in addition satisfies the following property: for any $n \geqslant 3$, there is in this family a walk associated with a reflection group of order $2n$. Moreover, the case $n = 4$ corresponds to a process which appears naturally by studying quantum random walks on the dual of Sp(4). For all the processes belonging to this family, we find the exact asymptotic of the Green functions along all infinite paths of states as well as that of the absorption probabilities along the boundaries.

Résumé

Dans cet article, nous considérons une famille de marches aléatoires tuées au bord de la chambre de Weyl du dual de Sp(4), qui vérifie en outre la propriété suivante : pour tout $n \geqslant 3$, il y a, dans cette famille, une marche ayant un groupe de réflexions d'ordre $2n$. De plus, le cas $n = 4$ correspond à un processus bien connu apparaissant lors de l'étude des marches aléatoires quantiques sur le dual de Sp(4). Pour tous les processus de cette famille, nous trouvons l'asymptotique exacte des fonctions de Green selon toutes les trajectoires, ainsi que l'asymptotique des probabilités d'absorption sur le bord.

Keywords: killed random walk, Green functions, Martin boundary, absorption probabilities

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1 Introduction and main results

Appearing in several distinct domains, random walks conditioned on staying in cones of $\mathbb{Z}^d$ attract more and more attention from the mathematical community. Historically, important examples are the so-called non-colliding random walks. These are the processes $(Z_1, \ldots, Z_d)$ composed of d independent and identically distributed random walks conditioned on never leaving the Weyl chamber $\{z \in \mathbb{R}^d : z_1 < \cdots < z_d\}$. They first appeared in the eigenvalues description of important matrix-valued stochastic processes, see [11], and are recently again very much studied, see [12, 8, 21] and the references therein. Another important area where processes conditioned on never leaving cones of $\mathbb{Z}^d$ appear is that of quantum random walks, see e.g. [3, 4].

A usual way to condition random processes on staying in cones consists in using Doob h-transforms. These are functions which are harmonic, positive inside of the cone and equal to zero on its boundary—or equivalently harmonic and positive for the underlying killed processes. It is therefore natural to be interested in finding all positive harmonic functions for processes in cones of $\mathbb{Z}^d$ killed at the boundary, and more generally to compute the Martin

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compactification of such processes, that can e.g. be obtained from the exact asymptotic of the Green functions.

We briefly recall [10] that for a transient Markov chain with state space E, the Martin compactification of E is the smallest compactification $\widehat{E}$ of E for which the Martin kernels $y \mapsto k_y^x = G_y^x / G_y^{x_0}$ extend continuously—by G_y^x we mean the Green functions and we denote by x_0 a reference state. $\widehat{E} \setminus E$ is usually called the full Martin boundary. For $\alpha \in \widehat{E}$, $x \mapsto k_\alpha^x$ is clearly superharmonic; then $\partial_m E = \{\alpha \in \widehat{E} \setminus E : x \mapsto k_\alpha^x \text{ is minimal harmonic}\}$ is called the minimal Martin boundary—a harmonic function h is said minimal if $0 \leqslant \widetilde{h} \leqslant h$ with $\widetilde{h}$ harmonic implies $\widetilde{h} = ch$ for some constant c. Then, every superharmonic (resp. harmonic) function h can be written as $h(x) = \int_{\widehat{E}} k_y^x \mu(\mathrm{d}y)$ (resp. $h(x) = \int_{\partial_m E} k_y^x \mu(\mathrm{d}y)$), where μ is some finite measure, uniquely characterized in the second case above.

In this context, the case of walks in cones of $\mathbb{Z}^d$ spatially homogeneous in the interior, with *non-zero drift* and killed at the boundary has held a great and fruitful deal of attention.

For random walks on weight lattices in Weyl chambers of Lie groups, Biane [4] finds the minimal Martin boundary thanks to Choquet-Deny theory. Collins [7] obtains their Martin compactification.

In [23] we give a more detailed analysis for a certain class of walks in dimension $d = 2$. These are the random walks killed at the boundary of $\mathbb{Z}_+^2$, with non-zero jump probabilities to the eight nearest neighbors and having in addition a positive mean drift. The asymptotic of the Green functions along all infinite paths of states as well as that of the probabilities of absorption along the axes are computed for the walks in this class. However, the methods of complex analysis used in [23] apply in dimension $d = 2$ only.

Ignatiouk-Robert [15, 14], then Ignatiouk-Robert and Loree [16] find the Martin compactification of the random walks in $\mathbb{Z}_+ \times \mathbb{Z}^{d-1}$ and $\mathbb{Z}_+^d$ ($d \geqslant 2$), with non-zero drift and killed at the boundary. They make very general assumptions on the jump probabilities. The approach used there, based on large deviation techniques and Harnack inequalities, seems not to be powerful for studying the asymptotic of the Green functions. Furthermore, having a non-zero drift is an essential hypothesis in [15, 16, 14]. Last but not least, the results of [14] in the case $d \geqslant 3$ are conditioned by the fact of been able: "to identify the positive harmonic functions of a random walk on $\mathbb{Z}^d$ which has zero mean and is killed at the first exit from $\mathbb{Z}_+^d$; unfortunately, for $d \geqslant 2$, there are no general results in this domain" (see page 5 of [14]).

As may this open problem suggest, the results and methods dealing with the asymptotic of Green functions or even with the Martin compactifiction for random walks in domains of $\mathbb{Z}^d$ with *drift zero* and killed at the boundary are actually scarce, even for $d = 2$.

In [25] (resp. [22]), approximations of the Green functions for simple random walks killed at the boundary of balls of $\mathbb{Z}^d$ (resp. of certain more general sets of $\mathbb{Z}^2$) are computed by comparison with Brownian motion. Namely, the Green functions $G(B)_y^x$ of Brownian motion (resp. random walk) killed at the boundary B are related to the potential kernels (or the Green functions if $d \geqslant 3$) a_y^x of the non-killed Brownian motion (resp. random walk), e.g. via the "balayage formula", see Chapter 4 of [25]. These identities have the same form for Brownian motion and random walk [25]:

$$G(B)_y^x = -a_y^x + \mathbb{E}_x\left[a_y^{S_{\tau_B}}\right] + F(B)^x,$$

where S is the process, τ_B is the hitting time of the boundary B and where the rest $F(B)^x$ can be expressed in terms of τ_B and x, indeed see Chapter 4 of [25]. They are then compared term by term. For the comparison of potential kernels a_y^x of the non-killed processes, classical formulas may be used, like that stated in Chapter 4 of [25]; for the comparison of positions of these processes at time of absorption, strong approximations of random walks by Brownian motion [19, 20] as well as Beurling estimates [18, 24] are usually exploited.

This approach in particular requires rather precise estimates of the hitting time τ_B. It seems therefore difficult to use for models of walks in cones where τ_B is painful to analyze.

This is for example the case of the random walks in the half-plane considered in [29, 30] and of the random walks in the quarter-plane we shall study in this paper (for which Chapter F of [27] illustrates the complexity of hitting times). Remark 8 in Subsection 1.3 specifies other reasons why this method via comparison with Brownian motion seems not to lead to enough satisfactory results in the analysis of the random walks we shall consider here.

For random walks in the half-plane, Uchiyama [29, 30] finds the asymptotic of the Green functions from their trigonometric representations.

If the domain is a quarter-plane, the simplest case is the cartesian product of two killed one-dimensional simple random walks with mean zero. The Martin boundary then happens to be trivial [26]. Moreover, the exact asymptotic of the Green functions for these processes is computed in Chapter D of [27].

From a Lie group theory point of view, the previous case corresponds to the group product SU(2)×SU(2), which is associated with a *reducible* rank-2 root system, see [5]. We are then interested in the classical random walks that can be obtained from the construction made by Biane in [4]—namely, by restriction of the quantum walks, initially defined on non-commutative von Neumann algebras, to commutative subalgebras—starting from Lie groups associated with *irreducible* rank-2 roots systems, namely SU(3) and Sp(4). These are random walks on the lattices of Figure 1, spatially homogeneous in the interior and absorbed at the boundary. The one on the left, associated with SU(3), has three jump probabilities equal to 1/3; that on the right, related to Sp(4), has the same four jump probabilities 1/4.

Figure 1: Random walks in the Weyl chamber of the duals of SU(3) and Sp(4)

By obvious transformations of these lattices it is immediate that both are killed nearest neighbors random walks in the quarter-plane $\mathbb{Z}_+^2$ as below.

Figure 2: Random walks of Figure 1 can be viewed as random walks in $\mathbb{Z}_+^2$

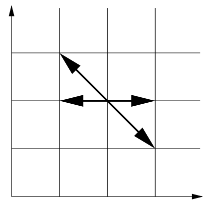
For the random walk associated with SU(3) (and besides also for its multi-dimensional analogues SU(d)), Biane [3] computes the asymptotic of the Green functions along all paths of states *except for the ones approaching the axes.* To complete the latter results for these particular paths, the complex analysis methods of [28] recently turned out to be fruitful. Furthermore, in [28] we compute the exact asymptotic of the Green functions along all paths and also the asymptotic of the absorption probabilities for a whole class of nearest neighbors random walks killed at the boundary of $\mathbb{Z}_+^2$. This class, including the walk associated with SU(3), is characterized by the fact that each of its elements has a harmonic function of order three, i.e. $(i_0, j_0) \mapsto i_0 j_0 (i_0 + \alpha j_0 + \beta)$ for some arbitrary α and β. Similar results for the simpler class of walks admitting the harmonic function of order two $(i_0, j_0) \mapsto i_0 j_0$ are derived in [27].

The methods of [28, 27] heavily rely on the analytic approach for the random walks in the quarter-plane developed by Fayolle, Iasnogorodski and Malyshev [13]. An important notion involved there is that of the *group of the random walk*, which is a group of automorphisms of an algebraic curve; it is here properly defined in Subsection 1.1. In this context our previous works [28, 27] treat the classes of walks with groups of the smallest orders four and six.

The main subject of this paper is to obtain, by this approach, the exact asymptotic of the Green functions along all paths as well as that of the absorption probabilities along the axes for a certain class of random walks in $\mathbb{Z}_+^2$ absorbed at the axes, with zero drift and containing the walk associated with Sp(4)—drawn on the right of Figure 2. This class will include walks with groups of all finite orders $2n$ for $n \geqslant 3$. We now define this class and specify the notion of group of the random walk.

1.1 Random walks under consideration and group of the walk

Consider the random walk $(X(k), Y(k))_{k\geqslant 0}$ spatially homogeneous inside of the quarter-plane $\mathbb{Z}_+^2$ and such that if $p_{i,j} = \mathbb{P}[(X(k+1), Y(k+1)) = (i_0+i, j_0+j) \mid (X(k), Y(k)) = (i_0, j_0)]$, then:



(H1) $p_{1,0} + p_{1,-1} + p_{-1,0} + p_{-1,1} = 1$, $p_{1,0} = p_{-1,0}$, $p_{1,-1} = p_{-1,1}$;

(H2) $\{(i,0) : \ i \geqslant 1\} \cup \{(0,j) : \ j \geqslant 1\}$ is absorbing.

Let us also define the polynomial Q (which is just a simple transformation of the transition probabilities generating function) by:

$$Q(x,y) = xy\big[p_{1,0}x + p_{-1,0}/x + p_{1,-1}x/y + p_{-1,1}y/x - 1\big]. \tag{1}$$

If $Q(x,y) = 0$, then with (1) it is immediate that $Q(\widehat{\xi}(x,y)) = 0$ and $Q(\widehat{\eta}(x,y)) = 0$, where

$$\widehat{\xi}(x,y) = \left(x, \frac{x^2}{y}\right), \qquad \widehat{\eta}(x,y) = \left(\frac{p_{-1,1}y + p_{-1,0}}{p_{1,0}y + p_{1,-1}}\frac{y}{x}, y\right).$$

The group of the walk is then $W = \langle \widehat{\xi}, \widehat{\eta} \rangle$, the group of automorphisms of the algebraic curve $\{(x,y) \in (\mathbb{C} \cup \{\infty\})^2 : Q(x,y) = 0\}$ generated by $\widehat{\xi}$ and $\widehat{\eta}$. Its order is always even and larger than or equal to four. It is already known [6] that if $p_{1,0} = p_{-1,0} = 1/4$ and $p_{1,-1} = p_{-1,1} = 1/4$ then W has order eight.

More generally, we prove in Remark 11 that the group W is finite if and only if there exists some rational number r such that $p_{1,0} = p_{-1,0} = \sin(r\pi)^2/2$ and $p_{1,-1} = p_{-1,1} = \cos(r\pi)^2/2$.

As illustrated by the works already mentioned [13, 6, 28, 27] the notion of finite group is nowadays extensively studied, notably because it often leads to worthwhile results. Let $\mathscr{P}_{2n}$ be the class of random walks satisfying (H1), (H2) and with a group of order $2n$. We show in Subsection 2.2 that the random walk under hypotheses (H1), (H2) and (H3), where

(H3) $p_{1,0} = p_{-1,0} = \sin(\pi/n)^2/2$ and $p_{1,-1} = p_{-1,1} = \cos(\pi/n)^2/2$,

belongs to $\mathscr{P}_{2n}$.

In this article we study the class being made up of the union for $n \geqslant 3$ of the random walks satisfying (H1), (H2) and (H3). This class contains one—and only one—representative of $\mathscr{P}_{2n}$ for any $n \geqslant 3$. Precisely, for all walks in this class we compute the exact asymptotic of the Green functions along all paths and that of the absorption probabilities along the axes. This is the first result of that kind for random walks with zero drift and groups of all finite orders, up to our knowledge.

In the particular case $n = 3$, the process coincides with that represented on the left of Figure 2 in [28] for $\alpha = 2$. In the case $n = 4$, this is the walk in the Weyl chamber of Sp(4) with jump probabilities 1/4 studied by Biane [4], see Figures 1 and 2.

The hypothesis that n is integer is technical. Indeed, independently of this assumption, the approach [13]—we shall use here—always yields explicit expressions for the Green functions, notably in terms of solutions to boundary value problems of Riemann-Hilbert type. In the general case, these formulations are so complex that we are not able to obtain their asymptotic. However, they may admit a nice simplification as a closed expression (and then in terms of the orbit under the group of a simple function); this actually happens if the walk admits a finite group, and if in addition some technical assumption holds—related to fundamental domains, see Subsection 2.2. As it will be properly showed in Remark 11, in the case of the walks satisfying to (H1) this exactly implies (H3).

1.2 Main results

Here and throughout, $(X,Y)=(X(k),Y(k))_{k\geqslant 0}$ denotes the process defined by (H1), (H2) and (H3). Our first result deals with the asymptotic of the Green functions, properly defined by

$$G_{i,j}^{i_0,j_0}=\mathbb{E}_{(i_0,j_0)}\left[\sum_{k\geqslant 0}\mathbf{1}_{\{(X(k),Y(k))=(i,j)\}}\right]. \tag{2}$$

Let f_n be the function defined in (23); in Section 3 we shall write it explicitly and we shall prove that it is harmonic for (X,Y), positive inside of $\mathbb{Z}_+^2$ and equal to zero on the boundary.

Theorem 1. *The Green functions (2) admit the following asymptotic as $i+j\to\infty$ and $j/i\to\tan(\gamma)$, $\gamma\in[0,\pi/2]$:*

$$G_{i,j}^{i_0,j_0}\sim\frac{2}{\pi}\frac{(n-1)!}{4^n\sin(2\pi/n)}f_n(i_0,j_0)\frac{\sin\left(n\arctan\left[\frac{j/i}{1+j/i}\tan(\pi/n)\right]\right)}{\left[\cos(\pi/n)^2\,(i^2+2ij)+j^2\right]^{n/2}}. \tag{3}$$

Remark 2. *Let $N_n(j/i)=\sin\left(n\arctan\left[\frac{j/i}{1+j/i}\tan(\pi/n)\right]\right)$ be the quantity appearing in the asymptotic (3). Let also γ be in $[0,\pi/2]$ and suppose that j/i goes to $\tan(\gamma)$.*

If $\gamma\in]0,\pi/2[$, then $N_n(j/i)$ goes to $N_n(\tan(\gamma))$, which belongs to $]0,\infty[$.

If $\gamma=0$ or $\gamma=\pi/2$, then $N_n(j/i)$ goes to 0. More precisely, $N_n(j/i)=n\tan(\pi/n)[j/i+O(j/i)^2]$ if $\gamma=0$ and $N_n(j/i)=(n\sin(2\pi/n)/2)[i/j+O(i/j)^2]$ if $\gamma=\pi/2$.

Theorem 1 has the following immediate [10] consequence.

Corollary 3. *The Martin compactification is the one-point compactification.*

This paper therefore gives a partial answer, for $d=2$, to the open problem highlighted by Ignatiouk-Robert in [14], since Corollary 3 implies that up to the positive multiplicative constants, there is only one positive harmonic function for (X,Y).

Theorem 1 also has a consequence on the absorption probabilities $\mathbb{P}_{(i_0,j_0)}[(X,Y)$ is killed at $(i,0)]$ and $\mathbb{P}_{(i_0,j_0)}[(X,Y)$ is killed at $(0,j)]$. Indeed, by using the obvious equalities

$$\begin{aligned}\mathbb{P}_{(i_0,j_0)}[(X,Y)\text{ is killed at }(i,0)]&=p_{1,-1}G_{i-1,1}^{i_0,j_0},\\ \mathbb{P}_{(i_0,j_0)}[(X,Y)\text{ is killed at }(0,j)]&=p_{-1,1}G_{1,j-1}^{i_0,j_0}+p_{-1,0}G_{1,j}^{i_0,j_0}\end{aligned}$$

as well as Theorem 1 and Remark 2, we come to the following result.

Corollary 4. *The absorption probabilities admit the following asymptotic as $i,j\to\infty$:*

$$\begin{aligned}\mathbb{P}_{(i_0,j_0)}[(X,Y)\textit{ is killed at }(i,0)]&\sim\frac{1}{2\pi}\frac{n!}{[4\cos(\pi/n)]^n}f_n(i_0,j_0)\frac{1}{i^{n+1}},\\ \mathbb{P}_{(i_0,j_0)}[(X,Y)\textit{ is killed at }(0,j)]&\sim\frac{1}{2\pi}\frac{n!}{4^n}f_n(i_0,j_0)\frac{1}{j^{n+1}}.\end{aligned}$$

1.3 Harmonic functions and link with Brownian motion

Let us now have a closer look at the harmonic function f_n that governs the asymptotic (3) of the Green functions (1). All the results of Subsection 1.3 are proven in Section 3.

Proposition 5.

(i) *f_n is a real polynomial in the variables i_0, j_0 of degree exactly n;*

(ii) *f_n is a harmonic function for the process (X, Y);*

(iii) *$f_n(i_0, 0) = f_n(0, j_0) = 0$ for all integers i_0 and j_0;*

(iv) *If $i_0, j_0 > 0$ then $f_n(i_0, j_0) > 0$.*

The explicit formulation of f_n for general values of n—that we shall obtain in Section 3—being quite complex, here we just give the following three examples:

$$\begin{aligned} f_3(i_0, j_0) &= 24 \cdot 3^{1/2} \cdot i_0 j_0 (i_0 + 2j_0), \\ f_4(i_0, j_0) &= (256/3) \cdot i_0 j_0 (i_0 + 2j_0)(i_0 + j_0), \\ f_6(i_0, j_0) &= (288/5)3^{1/2} \cdot i_0 j_0 (i_0 + 2j_0)(i_0 + j_0)\big((i_0 + 2j_0/3)(i_0 + 4j_0/3) + 10/9\big). \end{aligned}$$

Note that for $n = 3$ and $n = 4$, f_n is a homogeneous function of degree n (i.e. $f_n(\lambda x, \lambda y) = \lambda^n f_n(x, y)$), while f_6 is not. In fact the next result holds.

Proposition 6. *For any $n \geqslant 5$, f_n is not homogeneous.*

Let us conclude the introduction by outlining the link of f_n with the harmonic functions of Brownian motion. Let

$$\phi(x, y) = \big((x + y)/\sin(\pi/n), y/\cos(\pi/n)\big). \tag{4}$$

Then the random walk $\phi(X, Y)$ has an identity covariance and takes its values in the cone $\Lambda(0, \pi/n) = \{t \exp(\imath\theta) : 0 \leqslant t \leqslant \infty, 0 \leqslant \theta \leqslant \pi/n\}$.

Figure 3: On the left, the walk (X, Y); on the right, the walk $\phi(X, Y)$, with ϕ defined in (4)

$\phi(X, Y)$ therefore lies in the domain of attraction of the standard Brownian motion killed at the boundary of $\Lambda(0, \pi/n)$.

For the Brownian motion, it is well-known that there is only one harmonic function h positive inside of a given cone and vanishing on the boundary: it is called the réduite [1] of the cone. It happens to be homogeneous [1]. When the cone is $\Lambda(0, \pi/n)$, the réduite is equal to $h(\rho \exp(\imath\theta)) = \rho^n \sin(n\theta)$.

Moreover, the asymptotic of the Green functions of the Brownian motion killed at the boundary of $\Lambda(0, \pi/n)$ can be obtained from [2] and is equal to:

$$G_{r \exp(\imath\eta)}^{\rho \exp(\imath\theta)} \sim \frac{2}{\pi^{1/2}} h(\rho \exp(\imath\theta)) \frac{\sin(n\eta)}{r^n}, \qquad r \to \infty.$$

Proposition 7. *Let ϕ be defined in (4). Up to a multiplicative constant, the homogeneous function $h(\phi(i_0, j_0))$ equals the dominant term of the non-homogeneous harmonic function of the random walk, i.e.*

$$f_n(i_0, j_0) = h(\phi(i_0, j_0))[1 + o(1)], \qquad i_0, j_0 \to \infty.$$

Proposition 7 will follow from our results by a direct computation, see Section 3. Let us also note, as in [29], that this proposition is in accordance with Donsker's invariant principle.

Remark 8. *Proposition 7 also entails that the comparison approach of random walks by Brownian motion sketched in the first part of the introduction can neither give the approximation of the Green functions with the precision of Theorem 1 nor even to specify the unique harmonic function for our class of random walks (H1), (H2) and (H3).*

Furthermore, from the asymptotic results (3) we notice that the Green functions can tend to zero arbitrarily fast whenever the order of the group is taken high enough, while the Beurling estimates [18, 24] relating the random walk to the Brownian motion typically have a polynomial precision—which in addition depends only on the dimension.

The rest of the paper is organized as follows. In Section 2 we find explicitly the absorption probabilities and the Green functions (2). In Section 3, we then study precisely the harmonic function f_n. Finally in Section 4 we prove Theorem 1.

2 Expression of the absorption probabilities and of the Green functions

2.1 A functional equation between the generating functions

Subsection 2.1 consists in preparatory results and is inspired by the book [13]. Define

$$\begin{aligned} G^{i_0,j_0}(x,y) &= \sum_{i,j\geqslant 1} G_{i,j}^{i_0,j_0} x^{i-1} y^{j-1}, \\ h^{i_0,j_0}(x) &= \sum_{i\geqslant 1} \mathbb{P}_{(i_0,j_0)}[(X,Y) \text{ is killed at } (i,0)] x^i, \\ \widetilde{h}^{i_0,j_0}(y) &= \sum_{j\geqslant 1} \mathbb{P}_{(i_0,j_0)}[(X,Y) \text{ is killed at } (0,j)] y^j \end{aligned} \tag{5}$$

the generating functions of the Green functions (2) and of the absorption probabilities. With these notations, we can state the following functional equation:

$$Q(x,y)\, G^{i_0,j_0}(x,y) = h^{i_0,j_0}(x) + \widetilde{h}^{i_0,j_0}(y) - x^{i_0} y^{j_0}, \tag{6}$$

Q being defined in (1). A priori, Equation (6) has a meaning in $\{(x,y)\in\mathbb{C}^2 : |x|<1, |y|<1\}$. The proof of (6) is obtained exactly as in Subsection 2.1 of [23].

When no ambiguity on the initial state can arise, we will drop the index i_0, j_0 and we will write $G_{i,j}, G(x,y), h(x), \widetilde{h}(y)$ for $G_{i,j}^{i_0,j_0}, G^{i_0,j_0}(x,y), h^{i_0,j_0}(x), \widetilde{h}^{i_0,j_0}(y)$.

Let us now have a look to the algebraic curve $\{(x,y)\in(\mathbb{C}\cup\{\infty\})^2 : Q(x,y)=0\}$, that we note $\mathscr{Q}$ for the sake of briefness. Start by writing the polynomial (1) alternatively

$$Q(x,y) = a(x)\, y^2 + b(x)\, y + c(x) = \widetilde{a}(y)\, x^2 + \widetilde{b}(y)\, x + \widetilde{c}(y), \tag{7}$$

where $a(x) = p_{1,-1}$, $b(x) = p_{1,0}x^2 - x + p_{1,0}$, $c(x) = p_{1,-1}x^2$ and $\widetilde{a}(y) = p_{1,0}y + p_{1,-1}$, $\widetilde{b}(y) = -y$, $\widetilde{c}(y) = p_{1,-1}y^2 + p_{1,0}y$. Set also $d(x) = b(x)^2 - 4a(x)c(x)$ and $\widetilde{d}(y) = \widetilde{b}(y)^2 - 4\widetilde{a}(y)\widetilde{c}(y)$. We have

$$d(x) = p_{1,0}^2 (x-1)^2 \big(x^2 + 2x(1-1/p_{1,0}) + 1\big), \qquad \widetilde{d}(y) = -4p_{1,0}p_{1,-1}y(y-1)^2. \tag{8}$$

The polynomial d has manifestly a double root at 1 and two simple roots at positive points, that we denote by $x_1 < 1 < x_4$. As for $\widetilde{d}$, it has a double root at 1 and a simple root at 0. We also note $y_1 = 0$ and $y_4 = \infty$.

Then with (7) we notice that $Q(x, y) = 0$ is equivalent to $[b(x) + 2a(x)y]^2 = d(x)$ or to $[\widetilde{b}(y) + 2\widetilde{a}(y)x]^2 = \widetilde{d}(y)$. It follows from the particular form of d or $\widetilde{d}$, see (8), that the surface $\mathscr{Q}$ has genus zero and is thus homeomorphic to a sphere [17]. As a consequence this Riemann surface can be rationally uniformized, in the sense that it is possible to find two rational functions, say π and $\widetilde{\pi}$, such that $\mathscr{Q} = \{(\pi(s), \widetilde{\pi}(s)) : s \in \mathbb{C} \cup \{\infty\}\}$. Furthermore, as shown in Chapter 6 of [13], we can take $\pi(s) = [x_4+x_1]/2+([x_4-x_1]/4)(s+1/s)$, x_1 and x_4 being defined below (8); it is then possible to deduce a correct expression for $\widetilde{\pi}$, since by construction the equality $Q(\pi, \widetilde{\pi}) = 0$ has to hold. For more details about the construction of Riemann surfaces, see for instance [17].

2.2 Uniformization and meromorphic continuation

But rather than the uniformization $(\pi, \widetilde{\pi})$ proposed in [13] and recalled at the end of the previous subsection, we prefer using another, that will turn out to be quite more convenient. This new uniformization, that we call (x, y), is just equal to $(\pi \circ L, \widetilde{\pi} \circ L)$, where

$$L(z) = \frac{z_0 z - 1}{z - z_0}, \qquad z_0 = -\exp(-\imath\pi/n).$$

We notice that z_0 is such that $\pi(z_0) = \pi(\overline{z_0}) = \widetilde{\pi}(z_0) = \widetilde{\pi}(\overline{z_0}) = 1$ and that its explicit expression above is due to (H3), for more details see Remark 11. Then, starting from the formulations of $(\pi, \widetilde{\pi})$ and of L, we easily show that the expression of the new uniformization can be

$$x(z) = \frac{(z + z_0)(z + \overline{z_0})}{(z - z_0)(z - \overline{z_0})}, \qquad y(z) = \frac{(z + z_0)^2}{(z - z_0)^2}. \tag{9}$$

Compared to $(\pi, \widetilde{\pi})$, this uniformization (x, y) has the significant advantage of transforming the important cycles (i.e. the branch cuts $[x_1, x_4]$ and $[y_1, y_4]$, the unit circles $\{|x| = 1\}$ and $\{|y| = 1\}$) into very simple cycles, since the following equalities hold, see also Figure 4:

$$\begin{array}{rclrcl} x^{-1}([x_1, x_4]) & = & \mathbb{R} \cup \{\infty\}, & x^{-1}(\{|x| = 1\}) & = & \imath\mathbb{R} \cup \{\infty\}, \\ y^{-1}([y_1, y_4]) & = & z_0\mathbb{R} \cup \{\infty\}, & y^{-1}(\{|y| = 1\}) & = & z_0\imath\mathbb{R} \cup \{\infty\}. \end{array} \tag{10}$$

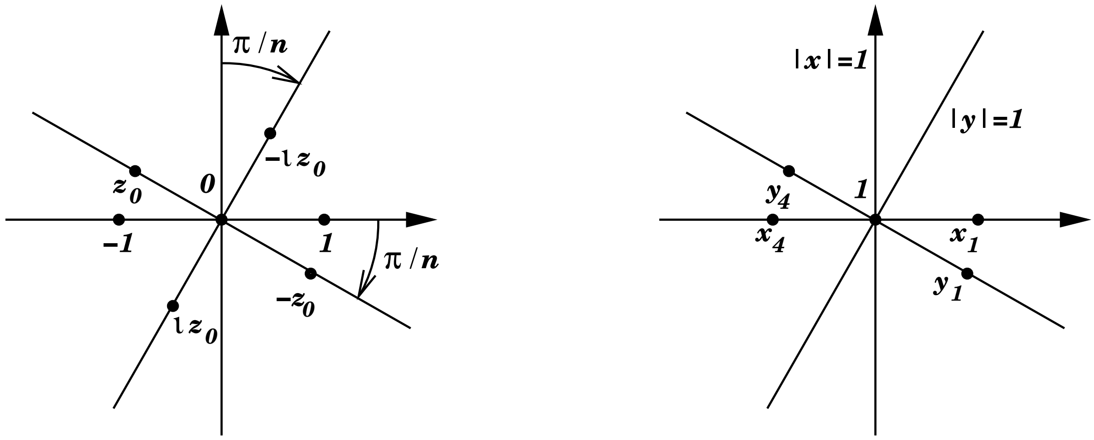


Figure 4: The uniformization space $\mathbb{C} \cup \{\infty\}$, with on the left some important elements of it and on the right the corresponding elements through the coordinates x and y

To obtain (10), it is sufficient to use the explicit expressions of the branch points x_1, x_4, y_1, y_4, see below (8), as well as the explicit formulation of the uniformization, see (9).

Let us go back to $\widehat{\xi}$ and $\widehat{\eta}$, the automorphisms of the algebraic curve $\mathscr{Q}$ introduced in Subsection 1.1. Thanks to the uniformization (9), they define two automorphisms ξ and η on $\mathbb{C} \cup \{\infty\}$, which are determined by

$$\xi^2 = 1, \quad x \circ \xi = x, \quad y \circ \xi = \frac{x^2}{y}, \quad \eta^2 = 1, \quad y \circ \eta = y, \quad x \circ \eta = \frac{p_{1,-1}y^2 + p_{1,0}y}{p_{1,0}y + p_{1,-1}}\frac{1}{x}. \tag{11}$$

Using the well-known characterization of the automorphisms of the Riemann sphere $\mathbb{C} \cup \{\infty\}$, (9) and (11), we obtain that ξ and η have the following expressions:

$$\xi(z) = 1/z, \qquad \eta(z) = \exp(-2\imath\pi/n)/z. \tag{12}$$

The expression above of η in terms of n is due to the assumption (H3), see Remark 11 for more details. Note also that leading to these particularly nice analytic expressions of ξ and η is another very pleasant property of the uniformization (x, y).

As in Subsection 1.1, we call the group generated by ξ and η

$$W_n = \langle \xi, \eta \rangle$$

the *group of the random walk.* In the context of this article, W_n is isomorphic to the dihedral group of order $2n$, i.e. to the group of symmetries of a regular polygon with n sides, ξ and η playing the role of the two reflections.

We are now going to state and prove Proposition 9, which actually is the main result of Subsection 2.2 and that deals with the continuation of the generating functions h and $\widetilde{h}$ defined in (5). For this, we need to describe the action of the elements of W_n on some cones of the plane as well as to find some fundamental domains of the plane for the action of W_n—we say that D is a fundamental domain of the plane for the action of W_n if $\cup_{w \in W_n} w(D) = \mathbb{C}$ and if in addition the latter union is disjoint.

Let us take the following notation: for $\theta_1 \leqslant \theta_2$, let

$$\Lambda(\theta_1, \theta_2) = \left\{ t \exp(\imath\theta) : 0 \leqslant t \leqslant \infty, \theta_1 \leqslant \theta \leqslant \theta_2 \right\}$$

be the cone with vertex at 0 and opening angles θ_1, θ_2. In particular, $\Lambda(\theta, \theta) = \exp(\imath\theta)\mathbb{R}_+ \cup \{\infty\}$. Thanks to (12), we obtain that the action of ξ and η on these cones is simply given by $\xi(\Lambda(\theta_1, \theta_2)) = \Lambda(-\theta_2, -\theta_1)$ and $\eta(\Lambda(\theta_1, \theta_2)) = \Lambda(-\theta_2 - 2\pi/n, -\theta_1 - 2\pi/n)$. These facts are illustrated on the left of Figure 5.

Define now, for $k \in \{0, \dots, n\}$,

$$D_k^+ = \Lambda\left(\frac{k-1}{n}\pi, \frac{k}{n}\pi\right), \qquad D_k^- = \Lambda\left(-\frac{k+1}{n}\pi, -\frac{k}{n}\pi\right).$$

Sometimes, we will write D_0 instead of $D_0^+ = D_0^-$ and D_n instead of $D_n^+ = D_n^-$. Clearly,

$$D_0 \cup D_n \cup \bigcup_{k=1}^{n-1} D_k^+ \cup \bigcup_{k=1}^{n-1} D_k^- = \mathbb{C} \cup \{\infty\}. \tag{13}$$

The definitions of D_k^+ and D_k^-, as well as (13), are illustrated on the right of Figure 5.

It is immediate that for any $k \in \{1, \dots, n\}$, we have $D_k^+ = \xi(D_{k-1}^-)$ and $D_k^- = \eta(D_{k-1}^+)$. In particular, for any $2k \in \{1, \dots, n\}$,

$$D_{2k}^+ = \big((\xi \circ \eta)^k\big)\big(D_0\big), \qquad D_{2k}^- = \big((\eta \circ \xi)^k\big)\big(D_0\big).$$

Likewise, for any $2k+1 \in \{1, \dots, n\}$,

$$D_{2k+1}^+ = \big(\xi \circ (\eta \circ \xi)^k\big)\big(D_0\big), \qquad D_{2k}^- = \big(\eta \circ (\xi \circ \eta)^k\big)\big(D_0\big).$$

With (13), these equalities prove that $\cup_{w \in W_n} w(D_0) = \mathbb{C} \cup \{\infty\}$, in such a way that D_0 is a fundamental domain for the action of W_n on $\mathbb{C}$—this is not quite exact, since each half-line $\Lambda(k\pi/n, k\pi/n)$, $k \in \{0, \dots, 2n-1\}$ appears twice in the union $\cup_{w \in W_n} w(D_0)$.

We are now able to state and prove Proposition 9, after the weak recall on the lifting of functions that follows: any function f of the variable x (resp. y) defined on some domain

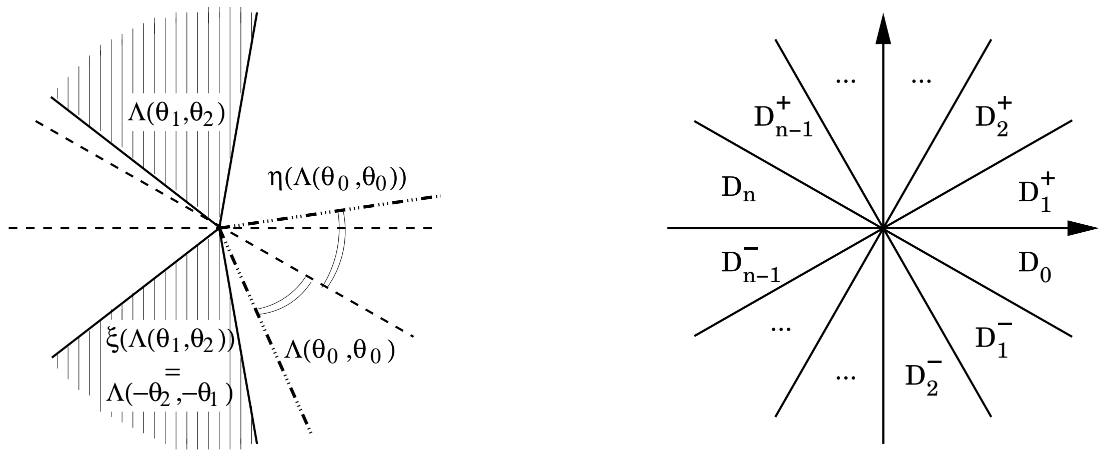


Figure 5: Important cones of the uniformization space

$D \subset \mathbb{C}$ can be lifted on $\{z \in \mathbb{C} \cup \{\infty\} : x(z) \in D\}$ (resp. $\{z \in \mathbb{C} \cup \{\infty\} : y(z) \in D\}$) by setting $F(z) = f(x(z))$ (resp. $F(z) = f(y(z))$).

In particular, we can lift the generating functions h and $\widetilde{h}$ considered in (5) and we set $H(z) = h(x(z))$ and $\widetilde{H}(z) = \widetilde{h}(y(z))$; they are well defined on $\{z \in \mathbb{C} \cup \{\infty\} : |x(z)| \leqslant 1\}$ and $\{z \in \mathbb{C} \cup \{\infty\} : |y(z)| \leqslant 1\}$ respectively. Consequently, on $\{z \in \mathbb{C} \cup \{\infty\} : |x(z)| \leqslant 1, |y(z)| \leqslant 1\}$ (which equals $\Lambda(-\pi/2, \pi/2 - \pi/n)$, see Figure 4), the functional equation (6) yields $H(z) + \widetilde{H}(z) - x^{i_0}y^{j_0}(z) = 0$—in the sequel, we will often write $x^{i_0}y^{j_0}(z)$ instead of $x(z)^{i_0}y(z)^{j_0}$.

Proposition 9. *The functions $H(z) = h(x(z))$ and $\widetilde{H}(z) = \widetilde{h}(y(z))$ can be meromorphically continued from respectively $\Lambda(-\pi/2, \pi/2)$ and $\Lambda(-\pi/2 - \pi/n, \pi/2 - \pi/n)$ up to respectively $\mathbb{C} \setminus \Lambda(\pi, \pi)$ and $\mathbb{C} \setminus \Lambda(\pi - \pi/n, \pi - \pi/n)$. Moreover, these continuations satisfy*

$$H(z) = H(\xi(z)), \qquad \widetilde{H}(z) = \widetilde{H}(\eta(z)), \qquad \forall z \in \mathbb{C}, \tag{14}$$

and

$$H(z) + \widetilde{H}(z) - x^{i_0}y^{j_0}(z) = \begin{cases} 0 & \text{if } z \notin \Lambda(\pi - \pi/n, \pi) \quad (15) \\ -\displaystyle\sum_{w \in W_n} (-1)^{l(w)}\, x^{i_0}y^{j_0}\,(w(z)) & \text{if } z \in \Lambda(\pi - \pi/n, \pi) \quad (16) \end{cases}$$

where $l(w)$ is the length of w, i.e. the smallest r for which we can write $w = w_1 \circ \cdots \circ w_r$, with $w_1, \ldots, w_r$ equal to ξ or η.

Remark 10. *As a consequence of Proposition 9, the generating functions h and $\widetilde{h}$ can be continued as meromorphic functions from the unit disc up to $\mathbb{C} \setminus [1, x_4]$ and $\mathbb{C} \setminus [1, y_4]$ respectively. Indeed, the formulas $h(x) = H(z)$ if $x(z) = x$ and $\widetilde{h}(y) = \widetilde{H}(z)$ if $y(z) = y$ define h and $\widetilde{h}$ not ambiguously, thanks to (14). Moreover, since we have $x(\Lambda(\pi, \pi)) = [1, x_4]$ and $y(\Lambda(\pi - \pi/n, \pi - \pi/n)) = [1, y_4]$, see (10) and Figure 4, the previous formulas yield meromorphic continuations on $\mathbb{C} \setminus [1, x_4]$ and $\mathbb{C} \setminus [1, y_4]$ respectively.*

Proof of Proposition 9. To prove Proposition 9, we shall heavily use the decomposition (13), and precisely, we are going to define H and $\widetilde{H}$ piecewise, by defining them on each of the $2n$ domains D that appear in the decomposition (13) to be equal to some functions H_D and $\widetilde{H}_D$; it will then suffice to show that the functions H and $\widetilde{H}$ so-defined satisfy the conclusions of Proposition 9.

$*$ In $D_0 \subset \{z \in \mathbb{C} \cup \{\infty\} : |x(z)| \leqslant 1, |y(z)| \leqslant 1\}$, we are going to use the most natural way to define H_{D_0} and $\widetilde{H}_{D_0}$, i.e. their power series; so for $z \in D_0$ we set $H_{D_0}(z) = h(x(z))$ and $\widetilde{H}_{D_0}(z) = \widetilde{h}(y(z))$.

$*$ Then, for $k \in \{1, \dots, n-1\}$, we define $H_{D_k^+}$, $\widetilde{H}_{D_k^+}$ on D_k^+ and $H_{D_k^-}$, $\widetilde{H}_{D_k^-}$ on D_k^- by

$$\begin{aligned}
\forall z \in D_k^+ = \xi\left(D_{k-1}^-\right): \quad & H_{D_k^+}(z) = H_{D_{k-1}^-}(\xi(z)), \quad \widetilde{H}_{D_k^+}(z) = -H_{D_k^+}(z) + x^{i_0}y^{j_0}(z), \\
\forall z \in D_k^- = \eta\left(D_{k-1}^+\right): \quad & \widetilde{H}_{D_k^-}(z) = \widetilde{H}_{D_{k-1}^+}(\eta(z)), \quad H_{D_k^-}(z) = -\widetilde{H}_{D_k^-}(z) + x^{i_0}y^{j_0}(z).
\end{aligned}$$

$*$ At last, for $z \in D_n$, we set $H_{D_n}(z) = H_{D_{n-1}^-}(\xi(z))$ and $\widetilde{H}_{D_n}(z) = \widetilde{H}_{D_{n-1}^+}(\eta(z))$.

Therefore we have, for each of the $2n$ domains D of the decomposition (13), defined two functions H_D and $\widetilde{H}_D$. Then, as said at the beginning of the proof, we set $H(z) = H_D(z)$ and $\widetilde{H}(z) = \widetilde{H}_D(z)$ for all $z \in D$ and for all domains D that appear in (13).

With this construction, (14) and (15) are immediately obtained. In order to prove (16), we can use the fact that it is possible to express *all* the functions H_D, $\widetilde{H}_D$ in terms of H_{D_0}, $\widetilde{H}_{D_0}$ and $x^{i_0}y^{j_0}$ *only*; we give, e.g., the expression of $H_{D_{2k}^+}$, for any $2k \in \{1, \dots, n\}$:

$$H_{D_{2k}^+}(z) = -\widetilde{H}_{D_0}\big((\eta\circ\xi)^k(z)\big) + \sum_{p=0}^{k-1} x^{i_0}y^{i_0}\big(\xi\circ(\eta\circ\xi)^p(z)\big) - \sum_{p=1}^{k-1} x^{i_0}y^{i_0}\big((\eta\circ\xi)^p(z)\big), \quad (17)$$

as well as that of $\widetilde{H}_{D_{2k}^-}$, for any $2k \in \{1, \dots, n\}$:

$$\widetilde{H}_{D_{2k}^-}(z) = -H_{D_0}\big((\xi\circ\eta)^k(z)\big) + \sum_{p=0}^{k-1} x^{i_0}y^{i_0}\big(\eta\circ(\xi\circ\eta)^p(z)\big) - \sum_{p=1}^{k-1} x^{i_0}y^{i_0}\big((\xi\circ\eta)^p(z)\big). \quad (18)$$

As a consequence, we get Equation (16) for even values of n. Indeed, for this it is actually sufficient first to add $H_{D_n}(z)$ and $\widetilde{H}_{D_n}(z)$, in other words the equalities (17) and (18) above for $k = n/2$, then to notice that if n is even, $(\xi\circ\eta)^{n/2} = (\eta\circ\xi)^{n/2}$, next to use that for $z \in D_n$, $H_{D_0}\big((\xi\circ\eta)^{n/2}(z)\big) + \widetilde{H}_{D_0}\big((\xi\circ\eta)^{n/2}(z)\big) = x^{i_0}y^{j_0}\big((\xi\circ\eta)^{n/2}(z)\big)$, see (15), and finally to notice that W_n equals

$$\big\{1, \eta\xi, \dots, (\eta\xi)^{n/2-1}, \xi\eta, \dots, (\xi\eta)^{n/2-1}, \xi, \dots, \xi(\eta\xi)^{n/2-1}, \eta, \dots, \eta(\xi\eta)^{n/2-1}, (\xi\eta)^{n/2}\big\}.$$

Likewise, we could write the expressions of

$$H_{D_{2k}^-}, \quad \widetilde{H}_{D_{2k}^+}, \quad H_{D_{2k+1}^+}, \quad \widetilde{H}_{D_{2k+1}^+}, \quad H_{D_{2k+1}^-}, \quad \widetilde{H}_{D_{2k+1}^-},$$

in terms of H_{D_0}, $\widetilde{H}_{D_0}$ and $x^{i_0}y^{j_0}$ and we would verify that Equation (16) is still true for odd n. Proposition 9 is proven. $\square$

Remark 11. *We can now explain precisely why the assumption* (H3) *dealing with the values of the transition probabilities is both natural and necessary for our study.*

If we suppose (H1) *but no more* (H3), *then the uniformization (9) is the same, with* $z_0 = -[2p_{1,-1}]^{1/2} + \imath[2p_{1,0}]^{1/2}$. *The transformations (10) of the important cycles through the uniformization are also still valid and the automorphism* ξ *is yet again equal to* $\xi(z) = 1/z$. *As for* η, *it equals* $\eta(z) = z_0^2/z$; *in particular, the group of the random walk* $W = \langle\xi, \eta\rangle$ *is finite if and only if there exists an integer* p *such that* $z_0^{2p} = 1$. *In this case, if* n *denotes the smallest of these positive integers* p, *the group* W *is then of order* $2n$.

If a such n *does not exist, then there is no hope to find a fundamental domain for the action of the group* W, *neither to obtain any equality like (16).*

If a such n *exists, then by using the fact that* $z_0^{2n} = 1$, *in other words the fact that* $(-[2p_{1,-1}]^{1/2} + \imath[2p_{1,0}]^{1/2})^{2n} = 1$, *we immediately obtain that* $p_{1,0} = \sin(q\pi/n)^2/2$, *for some integer* q *having a greatest common divisor with* n *equal to* 1.

In this last case, we have $z_0 = -\exp(-\imath q\pi/n)$; *it is then easily proven that the domain bounded by the cycles* $x^{-1}([1, x_4])$ *and* $y^{-1}([1, y_4])$, *namely* $\Lambda(\arg(z_0), \pi) = \Lambda(\pi - q\pi/n, \pi)$, *is a fundamental domain for the action of W if and only if $q = 1$. In particular, having an equality like (16) is possible if and only if $q = 1$, see the proof of Proposition 9.*

But it turns out that having an equality like (16) is essential in what follows, particularly in Section 4, where we have to know very precisely the behavior of $H(z) + \widetilde{H}(z) - x^{i_0}y^{j_0}(z)$ near 0 and ∞. In the general case, the formulations of H and $\widetilde{H}$—as solutions of boundary value problems—are so complex that we are not able to pursue the analysis.

For all these reasons, we assume here that $p_{1,0} = \sin(\pi/n)^2/2$ for some integer n, in other words nothing else but (H3).

3 Harmonic functions

Section 3 aims at introducing and studying a certain harmonic function associated with the process, which will be of the highest importance in the forthcoming Section 4.

It turns out that this harmonic function will be obtained from the expansion near 0 of $\sum_{w\in W_n}(-1)^{l(w)}x^{i_0}y^{j_0}(w(z))$, quantity which is appeared in (16); this is why we begin here by studying closely the behavior of the latter sum in the neighborhood of 0.

Note first that thanks to the expression (12) of the automorphisms ξ and η, we have

$$\sum_{w\in W_n} (-1)^{l(w)}\, x^{i_0}y^{j_0}\left(w\left(z\right)\right) = \sum_{k=0}^{n-1} \left[x^{i_0}y^{i_0}\left(\exp(-2\imath k\pi/n)z\right) - x^{i_0}y^{i_0}\left(\exp(-2\imath k\pi/n)/z\right)\right]. \tag{19}$$

Let us now take the following notations for the expansion at 0 of the function $x^{i_0}y^{j_0}$:

$$x^{i_0}y^{i_0}\left(z\right) = \sum_{p\geqslant 0} \kappa_p\left(i_0, j_0\right) z^p, \tag{20}$$

and notice that with (9), we obtain that for z close to 0,

$$x^{i_0}y^{i_0}\left(1/z\right) = \sum_{p\geqslant 0} \overline{\kappa_p}\left(i_0, j_0\right) z^p. \tag{21}$$

In a general setting, if f is holomorphic in a neighborhood of 0 with expansion $f(z) = \sum_{p\geqslant 0} f_p z^p$, then $\sum_{k=0}^{n-1} f(\exp(-2\imath k\pi/n)z) = \sum_{k=0}^{n-1} f(\exp(2\imath k\pi/n)z) = \sum_{p\geqslant 0} n f_{np} z^{np}$.

This is why, by using (20) and (21), we obtain that the sum (19) is equal to

$$\sum_{w\in W_n} (-1)^{l(w)}\, x^{i_0}y^{j_0}\left(w\left(z\right)\right) = \sum_{p\geqslant 1} n\left[\kappa_{np}\left(i_0, j_0\right) - \overline{\kappa_{np}}\left(i_0, j_0\right)\right] z^{np}. \tag{22}$$

We are now going to be interested in the term corresponding to $p = 1$ in the sum (22), and we set

$$f_n\left(i_0, j_0\right) = n\frac{\kappa_n\left(i_0, j_0\right) - \overline{\kappa_n}\left(i_0, j_0\right)}{\left(-1\right)^n \imath}. \tag{23}$$

We are now going to prove Proposition 5, but before we study some of its consequences.

Corollary 12. *The Doob f_n-transform process of (X, Y) never hits the boundary.*

Remark 13. *The function f_n defined in (23) is quite explicit. Indeed, by using the Cauchy product, κ_n (and therefore also f_n via (23)) can be written in terms of the coefficients of the expansions of x and y at 0, and these coefficients are easily calculated, see Equation (25). We don't know the factorized expression of f_n for general values of n. However, we have given in Subsection 1.3, as examples, the factorized forms of f_3, f_4 and f_6.*

Remark 14. *The explicit expression and the harmonicity of the function f_4 have already been obtained by Biane in [4].*

The quantity $f_4(i_0, j_0)$ also appears as a multiplicative factor in the asymptotic tail distribution of the hitting time of the boundary of $\mathbb{Z}_+^2$ for the process (X, Y) associated with $n = 4$ and starting from the initial state (i_0, j_0). Indeed, in [9, 27], denoting by $\tau = \inf\{k \geqslant 0 : X(k) = 0$ or $Y(k) = 0\}$, it is proven that $\mathbb{P}_{(i_0,j_0)}[\tau > k] \sim C f_4(i_0, j_0)/k^2$, where $C > 0$.

In particular, we can specify Corollary 12 in the case $n = 4$. Indeed, using the following equality for $l < k$ (obtained from the strong Markov property of the process (X, Y)):

$$\mathbb{P}_{(i_0,j_0)}\left[(X(l), Y(l)) = (i, j) \,|\tau > k\right] = \mathbb{P}_{(i_0,j_0)}\left[(X(l), Y(l)) = (i, j)\right] \frac{\mathbb{P}_{(i,j)}[\tau > k - l]}{\mathbb{P}_{(i_0,j_0)}[\tau > k]},$$

the asymptotic of [9, 27] yields that the Doob f_4-transform process is equal in distribution to the limit as $k \to \infty$ of the process conditioned on the event $[\tau > k]$.

Remark 15. *Let $n \geqslant 3$. Proposition 5 gives that there exists at least one positive harmonic function for the process (X, Y). Corollary 3 entails that f_n is in fact the unique—up to the positive multiplicative constants—positive harmonic function for (X, Y).*

Proof of Proposition 5. The fact that f_n takes real values is immediate from its definition. For the rest of the proof of (i), we are going to use the following straightforward fact: for any $f(z) = 1 + \sum_{p\geqslant 1} f_{p,1} z^p$, note $1 + \sum_{p\geqslant 1} f_{p,i} z^p$ the expansion at 0 of $f(z)^i$; then $f_{p,i}$ is a polynomial of degree equal or less than p in i, with dominant term equal to $f_{1,1}^p i^p/p!$. In particular, $f_{p,i}$ is of degree exactly p if and only if $f_{1,1} \neq 0$.

In our case, it is immediate from (9) that $\kappa_1(1, 0) = -4\cos(\pi/n) \neq 0$ and $\kappa_1(0, 1) = -4\exp(\imath\pi/n) \neq 0$. This is why, for any non-negative integer p, $\kappa_p(i, 0)$ is a polynomial of degree p in i; likewise, $\kappa_p(0, j)$ is a polynomial of degree p in j. In particular, $\kappa_n(i, j) = \sum_{p=0}^{n} \kappa_p(0, j)\kappa_{n-p}(i, 0)$ is a polynomial in i, j of degree n, with dominant term equal to

$$\sum_{p=0}^{n} \frac{\kappa_1(0, 1)^p}{p!} j^p \frac{\kappa_1(1, 0)^{n-p}}{(n-p)!} i^{n-p}.$$

In this way, we obtain that f_n is a polynomial in i, j of degree n, with dominant term equal to (after simplification)

$$\frac{2^{2n+1}}{(n-1)!} \sum_{p=1}^{n-1} C_n^p \sin(p\pi/n) \cos(\pi/n)^{n-p} j^p i^{n-p}. \tag{24}$$

Assertion (i) follows then immediately.

To prove (ii), it is enough to show that κ_n is harmonic. To show that, start by using the obvious equality $x^{i-1}y^{j-1}(z)Q(x(z), y(z)) = 0$, which reads $x^i y^j(z) = p_{1,0}x^{i+1}y^j(z) + p_{1,0}x^{i-1}y^j(z) + p_{1,-1}x^{i+1}y^{j-1}(z) + p_{1,-1}x^{i-1}y^{j+1}(z)$. Then, (20) yields that

$$\sum_{p\geqslant 0} \left[\kappa_p(i, j) - p_{1,0}\kappa_p(i+1, j) - p_{1,0}\kappa_p(i-1, j) - p_{1,-1}\kappa_p(i+1, j-1) - p_{1,-1}\kappa_p(i-1, j+1)\right] z^p$$

is identically zero; this means that all the κ_p, $p \geqslant 0$ are harmonic, hence in particular κ_n.

In order to prove (iii), we need to know explicitly the expansions of x and y at 0. From (9), we immediately obtain these expansions:

$$x(z) = 1 + \frac{4}{\tan(\pi/n)} \sum_{p\geqslant 1} (-1)^p \sin(p\pi/n) z^p, \quad y(z) = 1 + 4\sum_{p\geqslant 1} (-1)^p p \exp(\imath p\pi/n) z^p. \tag{25}$$

We show now the first part of (iii), namely the fact that $f_n(i, 0) = 0$ for all non-negative integer i. As it can be noticed from (25), the coefficients of x are real. For this reason, for all integers i and p, $\kappa_p(i, 0)$ is also real and thus $f_p(i, 0) = 0$; in particular, $f_n(i, 0) = 0$.

As for the second part of (iii), namely the fact that for all $j \geqslant 0$, $f_n(0,j) = 0$, we prove that $\kappa_n(0,j)$ is real—however, it isn't true that for all j and p, $\kappa_p(0,j)$ is real.

In order to obtain $\kappa_p(0,j)$—that is, the pth coefficient of the Taylor series of $y(z)^j$—we add all the terms of the form $\kappa_{p_1}(0,1)\kappa_{p_2}(0,1) \times \cdots \times \kappa_{p_j}(0,1)$ with $p_1 + \cdots + p_j = p$, this is nothing else but the Cauchy's product of the j series $y(z)$. In other words, using (25), we add terms of the form $p_1 \times \cdots \times p_j (-1)^{p_1+\cdots+p_j} \exp(\imath[p_1 + \cdots + p_j]\pi/n)$. As a consequence, $\kappa_p(0,j)$ can be written as $\varphi_p(j)(-1)^p \exp(\imath p\pi/n)$, with $\varphi_p(j) > 0$ if $j > 0$.

In the particular case $p = n$, we obtain $\kappa_n(0,j) = -\varphi_n(j)(-1)^n$; $\kappa_n(0,j)$ is therefore real and, immediately, $f_n(0,j) = 0$.

We prove now (iv). With (25), it is clear that the sequence $\kappa_0(1,0), \ldots, \kappa_{n-1}(1,0)$ is alternating, in the sense that for all $p \in \{0, \ldots, n-1\}$, $(-1)^p \kappa_p(1,0) > 0$. In particular, it follows from general results on power series that the sequence $\kappa_0(i,0), \ldots, \kappa_{n-1}(i,0)$ is still alternating, for any $i > 0$.

In addition, by using the Cauchy's product of $x(z)^i$ and $y(z)^j$, we obtain that $\kappa_n(i,j) = \kappa_n(i,0) + \kappa_n(0,j) + \sum_{p=1}^{n-1} (-1)^p \exp(\imath p\pi/n)\varphi_p(j)\kappa_{n-p}(i,0)$. Then, by definition of $f_n(i,j)$ and by using the fact that $\kappa_n(i,0)$ and $\kappa_n(0,j)$ are real, we get

$$f_n(i,j) = 2n\,(-1)^n \sum_{p=1}^{n-1} (-1)^p \sin(p\pi/n)\,\varphi_p(j)\kappa_{n-p}(i,0).$$

But we have already proven that $\varphi_p(j) > 0$ if $j > 0$ and that $(-1)^{n-p}\kappa_{n-p}(i,0) > 0$ if $i > 0$; above, f_n is thus written as the sum of $n-1$ positive terms, and is, therefore, positive. $\square$

Proof of Proposition 7. Recall that $h(\rho \exp(\imath\theta)) = \rho^n \sin(n\theta)$, see Subsection 1.3. Setting $u = \rho\cos(\theta)$ and $v = \rho\sin(\theta)$ then yields:

$$h(u,v) = \sum_{p=0}^{(n-1)/2} C_n^{2p+1} (-1)^p u^{n-(2p+1)} v^{2p+1}.$$

Next we easily check that up to a multiplicative constant, $h(\phi(i_0,j_0))$ equals the dominant term of $f_n(i_0,j_0)$, namely $(2^{2n+1}/(n-1)!) \sum_{p=1}^{n-1} C_n^p \sin(p\pi/n)\cos(\pi/n)^{n-p} i_0^p j_0^{n-p}$, see (24). $\square$

Proof of Proposition 6. Since f_n is a polynomial of degree exactly n, see (i) of Proposition 5, it is clearly enough to prove that for $n \geqslant 5$, $f_n(2,2)/f_n(1,1) \neq 2^n$. For this we shall find both $f_n(1,1)$ and $f_n(2,2)$ in terms of n, it will then be manifest that $f_n(2,2)/f_n(1,1) \neq 2^n$.

Let us first prove that $f_n(1,1) = 8n^3/\tan(\pi/n)$. The Cauchy product of $x(z)$ by $y(z)$ and the use of (25) yields

$$\kappa_n(1,1) = 4(-1)^{n+1}n + [16(-1)^n/\tan(\pi/n)] \sum_{p=1}^{n-1} p \exp(\imath p\pi/n) \sin((n-p)\pi/n).$$

Using then the exponential expression of $\sin((n-p)\pi/n)$ as well as the identity $\sum_{p=1}^{n-1} px^p = [nx^n(x-1) - x(x^n-1)]/[(x-1)^2]$ applied to $x = \exp(2\imath\pi/n)$ entails:

$$\kappa_n(1,1) = 4(-1)^{n+1}n + [8(-1)^n/(\imath\tan(\pi/n))][-n(n-1)/2 + n\exp(-\imath\pi/n)/(2\imath\sin(\pi/n))].$$

Using (23) finally gives the announced value of $f_n(1,1)$.

Thanks to calculations of the same kind than for $f_n(1,1)$, we reach the conclusion that $f_n(2,2) = (16/3)n^3[n^2 + 2 - 6/\tan(\pi/n)^2]/[\sin(\pi/n)^2\tan(\pi/n)]$. Then it becomes obvious that for $n \geqslant 5$, $f_n(2,2)/f_n(1,1) \neq 2^n$, which completes the proof of Proposition 6. $\square$

4 Asymptotic of the Green functions

Sketch of the proof of Theorem 1. We shall begin by expressing $G_{i,j}$ as a double integral, using for this Cauchy's formulas and (6), see Equation (26). Then we will make the change of variable given by the uniformization (9) and we will apply the residue theorem; in this way, we will write $G_{i,j}$ as the sum $G_{i,j,1}+G_{i,j,2}$ of two single integrals w.r.t. the uniformization variable but on two contours a priori different, see (27) and (28). Then we will show, using Cauchy's theorem and Proposition 9, that it is possible to move these contours of integration until having the same contours for both integrals $G_{i,j,1}$ and $G_{i,j,2}$. Finally, by using (16) we will obtain (29), which is the most important explicit formulation of the $G_{i,j}$, starting from which we will get their asymptotic. In (29), $G_{i,j}$ will be written as an integral on the contour $\exp(\imath\theta)\mathbb{R}_+\cup\{\infty\}$, for some $\theta\in[\pi-\pi/n,\pi]$.

After having chosen an appropriate value of $\theta\in[\pi-\pi/n,\pi]$, see (31), we will see that this is quite normal to decompose the contour into three parts, namely a neighborhood of 0, one of ∞ and an intermediate part. Indeed, the function $x(z)^i y(z)^j$ that appears in the integrand of (29) is, on the contour $\exp(\imath\theta)\mathbb{R}_+\cup\{\infty\}$, close to 1 near $0,\infty$ and strictly larger than 1 elsewhere. Next, we will study successively these contributions in three paragraphs, using for this essentially the Laplace's method, what will conclude the proof of Theorem 1.

Beginning of the proof of Theorem 1. Equation (6) yields immediately that the generating function G of the Green functions is holomorphic in $\{(x,y)\in\mathbb{C}^2:|x|<1,|y|<1\}$. As a consequence and using again Equation (6), the Cauchy's formulas allow us to write its coefficients $G_{i,j}$ as the following double integrals:

$$G_{i,j}=\frac{1}{[2\pi\imath]^2}\iint_{\substack{|x|=1\\|y|=1}}\frac{G(x,y)}{x^iy^j}\mathrm{d}x\mathrm{d}y=\frac{1}{[2\pi\imath]^2}\iint_{\substack{|x|=1\\|y|=1}}\frac{h(x)+\widetilde{h}(y)-x^{i_0}y^{j_0}}{x^iy^jQ(x,y)}\mathrm{d}x\mathrm{d}y, \tag{26}$$

where the circles $\{|x|=1\}=\{|y|=1\}=\{\exp(\imath\theta):\theta\in[0,2\pi[\}$ are orientated according to the increasing values of θ.

With (26), we can thus write $G_{i,j}$ as the sum $G_{i,j}=G_{i,j,1}+G_{i,j,2}$, where

$$G_{i,j,1}=\frac{1}{[2\pi\imath]^2}\int_{|x|=1}\frac{h(x)}{x^i}\int_{|y|=1}\frac{\mathrm{d}y}{y^jQ(x,y)}\mathrm{d}x,$$

$$G_{i,j,2}=\frac{1}{[2\pi\imath]^2}\int_{|y|=1}\frac{\widetilde{h}(y)}{y^j}\int_{|x|=1}\frac{\mathrm{d}x}{x^iQ(x,y)}\mathrm{d}y+\frac{1}{[2\pi\imath]^2}\int_{|y|=1}\frac{1}{y^{j-j_0}}\int_{|x|=1}\frac{\mathrm{d}x}{x^{i-i_0}Q(x,y)}\mathrm{d}y.$$

We are now going to make in $G_{i,j,1}$ the change of variable $x=x(z)$. For this, we notice that if $\Lambda(\pi/2,\pi/2)=\{\imath t:t\in[0,\infty]\}$ is orientated according to increasing values of t, then the equality $x(\Lambda(\pi/2,\pi/2))=-\{|x|=1\}$ holds—in the sense of the orientated contours—, see (10) and Figure 4. In this way and using in addition the identity $q(x(z))=H(z)$, we get

$$G_{i,j,1}=-\frac{1}{[2\pi\imath]^2}\int_{\Lambda(\pi/2,\pi/2)}\frac{H(z)}{x(z)^i}\int_{|y|=1}\frac{\mathrm{d}y}{y^jQ(x(z),y)}x'(z)\mathrm{d}z.$$

But $Q(x(z),y)=0$ if and only if $y\in\{y(z),x(z)^2/y(z)\}$, see (11). Moreover, if z belongs to $\Lambda(\pi/2,\pi/2)\setminus\{0,\infty\}$, then $|y(z)|>1$, see Figure 4. The residue theorem at infinity therefore entails that for such z, $\int_{|y|=1}\mathrm{d}y/[y^jQ(x(z),y)]=-2\pi\imath/[y(z)^j\partial_yQ(x(z),y(z))]$. Finally, we have proven that

$$G_{i,j,1}=\frac{1}{2\pi\imath}\int_{\Lambda(\pi/2,\pi/2)}\frac{H(z)}{x(z)^i\,y(z)^j}\frac{x'(z)}{\partial_yQ(x(z),y(z))}\mathrm{d}z. \tag{27}$$

A similar reasoning yields

$$G_{i,j,2}=-\frac{1}{2\pi\imath}\int_{\Lambda(-\pi/2-\pi/n,-\pi/2-\pi/n)}\frac{\widetilde{H}(z)-x(z)^{i_0}\,y(z)^{j_0}}{x(z)^i\,y(z)^j}\frac{y'(z)}{\partial_xQ(x(z),y(z))}\mathrm{d}z. \tag{28}$$

We are now going to explain why it is possible to move the contours of integration of both integrals (27) and (28) up to $\Lambda(\theta,\theta)$, for any $\theta\in[\pi-\pi/n,\pi]$—see Figure 6 below.

Start by considering $G_{i,j,1}$ in (27). Thanks to Cauchy's theorem, it is sufficient to show that the integrand of $G_{i,j,1}$ is holomorphic inside of $\Lambda(\pi/2,\pi)$, domain which is horizontally hatched on Figure 6; let us thus prove this fact.

On one hand, with (1) and (9), on the domain $\Lambda(\pi/2,\pi)$ we get $x'(z)/\partial_y Q(x(z),y(z)) = -\imath/(2[p_{1,0}p_{1,-1}]^{1/2}z)$, and the latter function has manifestly no pole. On the other hand, it is possible to deduce from the proof of Proposition 9 that the only poles of H are at z_0 and $\overline{z_0}$. In particular, using (9), we obtain that for i or j large enough, $H(z)/[x(z)^i y(z)^j]$ has no pole in $\Lambda(\pi/2,\pi)$. Therefore, for i or j large enough, the integrand of $G_{i,j,1}$ has no pole in $\Lambda(\pi/2,\pi)$ and we can thus move the contour from $\Lambda(\pi/2,\pi/2)$ to $\Lambda(\theta,\theta)$, for any $\theta\in[\pi/2,\pi]$. Note that it isn't possible to move the contour beyond $\Lambda(\pi,\pi)$, since $\Lambda(\pi,\pi)$ is a singular curve for H—indeed, remember that $\Lambda(\pi,\pi)=x^{-1}([1,x_4])$ and see Proposition 9.

By similar considerations, we obtain that it is possible to move the initial contour of integration of $G_{i,j,2}$ up to $\Lambda(\theta,\theta)$, for any $\theta\in[\pi-\pi/n,3\pi/2-\pi/n]$.

In particular, if we wish to have the same contour of integration for $G_{i,j,1}$ and $G_{i,j,2}$, we can choose $\Lambda(\theta,\theta)$, for any $\theta\in[\pi-\pi/n,\pi]=[\pi/2,\pi]\cap[\pi-\pi/n,3\pi/2-\pi/n]$.

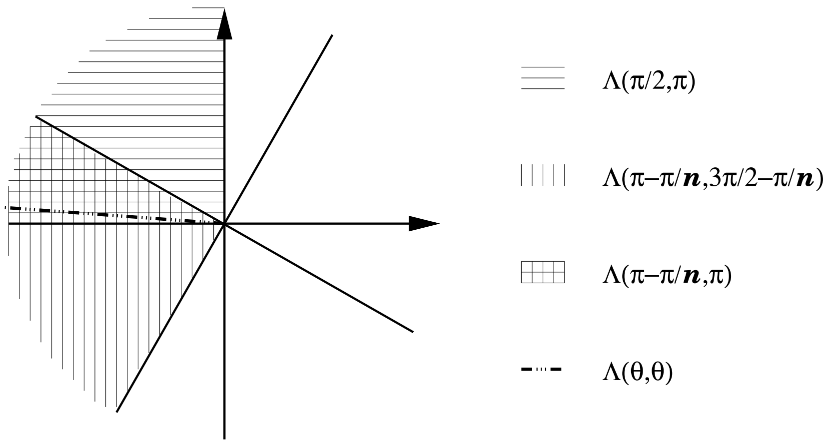


Figure 6: Change of the contours of integration in the integrals (27) and (28)

Using then the equality $x'(z)/\partial_y Q(x(z),y(z)) = -y'(z)/\partial_x Q(x(z),y(z))$, that comes from differentiating $Q(x(z),y(z))=0$, as well as (27), (28) and (16)—we can use (16) since $\theta\in[\pi-\pi/n,\pi]$ and thus $\Lambda(\theta,\theta)\subset\Lambda(\pi-\pi/n,\pi)$—, we obtain the following final explicit formulation for $G_{i,j}$, θ being any angle in $[\pi-\pi/n,\pi]$:

$$G_{i,j}=\frac{1}{4\pi[p_{1,0}p_{1,-1}]^{1/2}}\int_{\Lambda(\theta,\theta)}\left[\frac{1}{z}\sum_{w\in W_n}(-1)^{l(w)}\,x^{i_0}y^{j_0}\,(w\,(z))\right]\frac{1}{x\,(z)^i\,y\,(z)^j}\mathrm{d}z. \qquad (29)$$

On the contour $\Lambda(\theta,\theta)\subset\Lambda(\pi-\pi/n,\pi)$, the modulus of the function $x(z)^i y(z)^j$ is larger than 1, see Figure 4. Moreover, it goes to 1 when and only when z goes to 0 or to ∞. This is why it seems normal to decompose the contour $\Lambda(\theta,\theta)$ into a part near 0, another near ∞ and the residual part, and to think that the parts near 0 and ∞ will lead to the asymptotic of $G_{i,j}$, while the residual part will lead to a negligible contribution. But how to find the best contour in order to achieve this idea? In other words, how to find the value of $\theta\in[\pi-\pi/n,\pi]$ for which the calculation of the asymptotic of (29) will be the easiest?

For this, we are going to consider with details the function $x(z)^i y(z)^j$, or equivalently

$$\chi_{j/i}(z)=\ln(x(z))+(j/i)\ln(y(z)).$$

Incidentally this is why, from now on, we suppose that $j/i \in [0, M]$, for some $M < \infty$. Indeed, the function $\chi_{j/i}$ is manifestly not adapted to the values j/i going to ∞; for such j/i, we will consider, later, the function $(i/j)\chi_{j/i}(z) = (i/j)\ln(x(z)) + \ln(y(z))$. Nevertheless, M can be so large as wished and in what follows, we assume that some $M > 0$ is fixed.

With (9), we easily obtain the explicit expansion of $\chi_{j/i}$ at 0:

$$\chi_{j/i}(z) = \sum_{p\geqslant 0} \nu_{2p+1}(j/i)\, z^{2p+1}, \qquad \nu_{2p+1}(j/i) = \frac{2}{2p+1}\left[{z_0}^{2p+1} + \overline{z_0}^{2p+1} + 2\,(j/i)\,\overline{z_0}^{2p+1}\right]. \tag{30}$$

Likewise, again with (9), we get that for z near ∞, $\chi_{j/i}(z) = \sum_{p=0}^{\infty} \overline{\nu_{2p+1}}(j/i)1/z^{2p+1}$.

Consider now the steepest descent path associated with $\chi_{j/i}$, in other words the function $z_{j/i}(t)$ defined by $\chi_{j/i}(z_{j/i}(t)) = t$. By inverting the latter equality, we easily obtain that the half-line $(1/\nu_1(j/i))\mathbb{R}_+ \cup \{\infty\}$ is tangent at 0 and at ∞ to this steepest descent path.

Let us now set

$$\rho_{j/i} = 1/\nu_1(j/i) = 1/\left[2\left(z_0 + \overline{z_0} + 2\,(j/i)\,\overline{z_0}\right)\right]. \tag{31}$$

With this notation, we now answer the question asked above, that dealt with the fact of finding the value of θ for which the asymptotic of the Green functions (29) will be the most easily calculated: we choose $\theta = \arg(\rho_{j/i})$—note that, from the definition of z_0 and (31), we immediately obtain that $\arg(\rho_{j/i}) \in [\pi - \pi/n, \pi]$—, and the decomposition of the contour $\Lambda(\theta, \theta)$ is

$$\Lambda\left(\arg(\rho_{j/i}), \arg(\rho_{j/i})\right) = \left(\rho_{j/i}/|\rho_{j/i}|\right)[0,\epsilon] \cup \left(\rho_{j/i}/|\rho_{j/i}|\right)]\epsilon, 1/\epsilon[\cup \left(\rho_{j/i}/|\rho_{j/i}|\right)[1/\epsilon, \infty].$$

According to this decomposition and to (29), we consider now $G_{i,j}$ as the sum of three terms and we are going to study successively the contribution of each of these terms.

***Contribution of the neighborhood of* 0.** In order to evaluate the asymptotic of the integral (29) on the contour $(\rho_{j/i}/|\rho_{j/i}|)[0,\epsilon]$, we are going to use the expansion at 0 of the function

$$\frac{1}{z}\sum_{w\in W_n} (-1)^{l(w)}\, x^{i_0} y^{j_0}\,(w(z)).$$

This is why we begin here by studying the asymptotic of the following integral:

$$\int_{(\rho_{j/i}/|\rho_{j/i}|)[0,\epsilon]} \frac{z^k}{x(z)^i\, y(z)^j}\mathrm{d}z, \tag{32}$$

k being some non-negative integer. Using the equality $1/[x(z)^i y(z)^j] = \exp(-i\chi_{j/i}(z))$ as well as the expansion (30) of $\chi_{j/i}$ at 0 and then making the change of variable $z = \rho_{j/i}t$, we obtain that (32) is equal to

$$\rho_{j/i}^{k+1}\int_0^{\epsilon/|\rho_{j/i}|} t^k \exp(-it)\exp\left(-i\sum_{p\geqslant 1}\nu_{2p+1}(j/i)\left(\rho_{j/i}t\right)^{2p+1}\right)\mathrm{d}t. \tag{33}$$

But thanks to (30), $|\nu_{2p+1}(j/i)| \leqslant 4(M+1)$ and therefore for all $t \in [0, \epsilon/|\rho_{j/i}|]$, we have $|-i\sum_{p=1}^{\infty}\nu_{2p+1}(j/i)(\rho_{j/i}t)^{2p+1}| \leqslant i\epsilon^3 4(M+1)/(1-\epsilon^2)$. This is why

$$\exp\left(-i\sum_{p\geqslant 1}\nu_{2p+1}(j/i)\left(\rho_{j/i}t\right)^{2p+1}\right) = 1 + O\left(i\epsilon^3\right),$$

the O above being independent of $j/i \in [0, M]$ and of $t \in [0, \epsilon/|\rho_{j/i}|]$. The integral (33) can thus be calculated as

$$\rho_{j/i}^{k+1}\left[1 + O\left(i\epsilon^3\right)\right]\int_0^{\epsilon/|\rho_{j/i}|} t^k \exp(-it)\mathrm{d}t = \left(\rho_{j/i}/i\right)^{k+1}\left[1 + O\left(i\epsilon^3\right)\right]\int_0^{i\epsilon/|\rho_{j/i}|} t^k \exp(-t)\,\mathrm{d}t.$$

In the sequel, we choose $\epsilon = 1/i^{3/4}$, so that $i\epsilon/|\rho_{j/i}| \to \infty$ and $O(i\epsilon^3) = O(1/i^{5/4})$.

One could be surprised by this choice of ϵ; in fact, in the upcoming paragraph "Conclusion", we will see that in order to obtain the asymptotic of the Green functions along the paths of states $(i,j) \in \mathbb{Z}_+^2$ such that $j/i \to \tan(\gamma) \in]0,\infty[$, it would have been sufficient to have $O(i\epsilon^3) = o(1)$, but for the paths $(i,j) \in \mathbb{Z}_+^2$ such that $j/i \to 0$, it is necessary to have $O(i\epsilon^3) = o(1/i)$, what affords the choice $\epsilon = 1/i^{3/4}$.

Finally, we obtain that for this choice of ϵ, the integral (32) is equal to

$$\int_{(\rho_{j/i}/|\rho_{j/i}|)[0,\epsilon]} \frac{z^k}{x\,(z)^i\, y\,(z)^j} \mathrm{d}z = \left(\rho_{j/i}/i\right)^{k+1} k! \left[1 + O\left(1/i^{5/4}\right)\right], \tag{34}$$

where the O is independent of $j/i \in [0, M]$.

We are presently ready to obtain the asymptotic of the integral (29) on the contour $(\rho_{j/i}/|\rho_{j/i}|)[0,\epsilon]$. First, in accordance with (22), we have that this integral equals

$$\frac{1}{4\pi[p_{1,0}p_{1,-1}]^{1/2}} \sum_{p\geqslant 1} n \left[\kappa_{np}\,(i_0,j_0) - \overline{\kappa_{np}}\,(i_0,j_0)\right] \int_{(\rho_{j/i}/|\rho_{j/i}|)[0,\epsilon]} \frac{z^{np-1}}{x\,(z)^i\, y\,(z)^j} \mathrm{d}z.$$

Thus clearly, with (34), we obtain that all the terms corresponding in the sum above to $p \geqslant 2$ will be negligible w.r.t. the one associated with $p = 1$. In addition, by using the definition (23) of the harmonic function f_n as well as (34) for $k = pn - 1$ and $p \geqslant 1$, we get that the integral (29) on the contour $(\rho_{j/i}/|\rho_{j/i}|)[0,\epsilon]$ is equal to

$$\frac{1}{4\pi[p_{1,0}p_{1,-1}]^{1/2}} (-1)^n\,(n-1)! f_n\,(i_0,j_0)\,\imath \left(\rho_{j/i}/i\right)^n \left[1 + O\left(1/i^{5/4}\right)\right]. \tag{35}$$

Contribution of the neighborhood of ∞***.*** The part of the contour close to ∞, namely $(\rho_{j/i}/|\rho_{j/i}|)[1/\epsilon,\infty]$, is related to the part $(\rho_{j/i}/|\rho_{j/i}|)[0,\epsilon]$ via the transformation $z \mapsto 1/\overline{z}$. Moreover, it is clear from (9) that for $f = x$, $f = y$ or $f = \sum_{w\in W_n}(-1)^{l(w)} x^{i_0} y^{j_0}(w)$,

$$f\,(1/\overline{z}) = \overline{f\,(z)}.$$

Therefore, the change of variable $z \mapsto 1/\overline{z}$ immediately gives us that the contribution of the integral (29) near ∞ is the complex conjugate of its contribution near 0.

Contribution of the intermediate part. Let A_ϵ be the annular domain $\{z \in \mathbb{C} : \epsilon \leqslant |z| \leqslant 1/\epsilon\}$. According to Figure 4, for all $z \in \Lambda(\pi - \pi/n, \pi) \cap A_\epsilon$ we have $|x(z)| > 1 + \eta_{x,\epsilon}$ and $|y(z)| > 1 + \eta_{y,\epsilon}$, where $\eta_{x,\epsilon} > 0$ and $\eta_{y,\epsilon} > 0$. In fact, since $x'(0) \neq 0$ and $y'(0) \neq 0$, we can take $\eta_{x,\epsilon} > \eta\epsilon$ and $\eta_{y,\epsilon} > \eta\epsilon$ for some $\eta > 0$ independent of ϵ small enough.

Let us now consider

$$L = \sup_{z\in\Lambda(\pi-\pi/n,\pi)} \left| \left[\sum_{w\in W_n} (-1)^{l(w)}\, x^{i_0} y^{j_0}\,(w\,(z)) \right] \Big/ \left[x^{i_0} y^{j_0}\,(z) \right] \right|,$$

and let us show that L is finite. For this, it is enough to prove that the function s, defined by $s(z) = \left[\sum_{w\in W_n}(-1)^{l(w)} x^{i_0} y^{j_0}(w(z))\right] / \left[x^{i_0} y^{j_0}(z)\right]$, has no pole in $\Lambda(\pi - \pi/n, \pi)$—including ∞. But by using (9) and (12), we see that the only poles of the numerator of s are the $z_0 \exp(2\imath p\pi/n)$ for $p \in \{0, \ldots, n-1\}$. Among these n points, only z_0 is in $\Lambda(\pi - \pi/n, \pi)$. But in s, we have taken care of dividing by $x^{i_0} y^{j_0}(z)$, so that s is in fact holomorphic near z_0. Moreover, it is easily shown that s is holomorphic at ∞. Finally, we have proven that s has no pole in $\Lambda(\pi - \pi/n, \pi)$, hence s is bounded in $\Lambda(\pi - \pi/n, \pi)$; in other words, L is finite.

The modulus of the contribution of (29) on the intermediate part $(\rho_{j/i}/|\rho_{j/i}|)]\epsilon, 1/\epsilon[\subset \Lambda(\pi-\pi/n,\pi)\cap A_\epsilon$ can therefore be bounded from above by

$$\frac{1}{4\pi[p_{1,0}p_{1,-1}]^{1/2}}\frac{1}{\epsilon^2}\frac{L}{(1+\eta\epsilon)^{i-i_0}(1+\eta\epsilon)^{j-j_0}}. \tag{36}$$

Note that the presence of the term $1/\epsilon^2$ in (36) is due to the following: one $1/\epsilon$ appears as an upper bound of the length of the contour, while the other $1/\epsilon$ comes from an upper bound of the modulus of the term $1/z$ present in the integrand of (29).

Then as before we take $\epsilon = 1/i^{3/4}$, and then we use the following straightforward upper bound, valid for i large enough: $1/[1+\eta/i^{3/4}]^i \leqslant \exp(-[\eta/2]i^{1/4})$. We finally obtain that for i large enough, (36) is equal to $O(i^{3/2}\exp(-[\eta/2]i^{1/4}))$.

Conclusion. We have seen that the contribution of the integral (29) in the neighborhood of 0 is given by (35), that the contribution of (29) in the neighborhood of ∞ is equal to the complex conjugate of (35) and that the contribution of the residual part can be written as $O(i^{3/2}\exp(-[\eta/2]i^{1/4})$. Therefore, with (29) and (35), we obtain that

$$G_{i,j} = \frac{1}{4\pi[p_{1,0}p_{1,-1}]^{1/2}}(-1)^n(n-1)!f_n(i_0,j_0)\,\imath\left[\left(\rho_{j/i}/i\right)^n - \left(\overline{\rho_{j/i}}/i\right)^n\right] + O\big(1/i^{n+5/4}\big). \tag{37}$$

Moreover, starting from (31), we easily derive

$$\left(\rho_{j/i}/i\right)^n - \left(\overline{\rho_{j/i}}/i\right)^n = \frac{2\imath\,(-1)^{n+1}}{4^n}\frac{\sin\left(n\arctan\left(\frac{j/i}{1+j/i}\tan(\pi/n)\right)\right)}{\left[\cos(\pi/n)^2\,(i^2+2ij)+j^2\right]^{n/2}}.$$

The latter equality, (37) and Remark 2 conclude the proof of Theorem 1 in the case of $\gamma\in[0,\pi/2[$.

* Note that having $o(1/i^n)$ instead of $O(1/i^{n+5/4})$ would have been sufficient for $\gamma\in\,]0,\pi/2[$, since in this case, Remark 2 implies that $(\rho_{j/i}/i)^n - (\overline{\rho_{j/i}}/i)^n \sim K_\gamma/i^n$ with $K_\gamma\neq 0$.
* On the other hand, if $\gamma=0$ then $(\rho_{j/i}/i)^n - (\overline{\rho_{j/i}}/i)^n \sim K_0 j/i^{n+1}$ with $K_0\neq 0$ and it is necessary to have something like $o(1/i^{n+1})$ in (37), as it is actually the case with $O(1/i^{n+5/4})$.

To prove Theorem 1 in the case $\gamma=\pi/2$, we would consider $(i/j)\kappa_{j/i}$ rather than $\kappa_{j/i}$ and we would then use exactly the same analysis; we omit the details.

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